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2005 J. Phys. A: Math. Gen. 38 3947

(http://iopscience.iop.org/0305-4470/38/18/006)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 3947–3964 doi:10.1088/0305-4470/38/18/006

On a relation of pseudoanalytic function theory to the two-dimensional stationary Schrödinger equation and Taylor series in formal powers for its solutions

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Received 20 January 2005
Published 18 April 2005
Online at stacks.iop.org/JPhysA/38/3947

Abstract
We consider the real stationary two-dimensional Schrödinger equation. With the aid of any of its particular solutions, we construct a Vekua equation possessing the following special property. The real parts of its solutions are solutions of the original Schrödinger equation and the imaginary parts are solutions of an associated Schrödinger equation with a potential having the form of a potential obtained after the Darboux transformation. Using Bers' theory of Taylor series for pseudoanalytic functions, we obtain a locally complete system of solutions of the original Schrödinger equation which can be constructed explicitly for an ample class of Schrödinger equations. For example it is possible when the potential is a function of one Cartesian, spherical, parabolic or elliptic variable. We give some examples of application of the proposed procedure for obtaining a locally complete system of solutions of the Schrödinger equation. The procedure is algorithmically simple and can be implemented with the aid of a computer system of symbolic or numerical calculation.

PACS numbers: 02.30.−f, 02.30.Tb

1. Introduction

The appearance of a mathematical theory depends a lot on the preferences and mathematical tastes of its creator. Different people regard and describe the same mathematical results and ideas from different viewpoints, and this is one of the sources of the richness and progress of our science.

The pseudoanalytic function theory is one of the clearest confirmations of this assertion. It was independently developed by two prominent mathematicians, I N Vekua and L Bers, with coauthors, and presented in their books [3] and [12]. The theory received further development in hundreds of subsequent works (see, e.g., the reviews [2, 11]), and historically it became one of the important impulses for developing the general theory of elliptic systems. Here, the Vekua theory played a more important role due to its tendency to a more general, operational approach. Bers tried to follow more closely the ideas of classical complex analysis and paid more attention to the efficient construction of solutions. Among other results, Bers obtained analogues of the Taylor series for pseudoanalytic functions and some recursion formulae for constructing generalizations of the base system $1, z, z^2, \ldots$. The formulae require knowledge of the Bers generating pair (two special solutions) of the corresponding Vekua equation describing pseudoanalytic functions as well as generating pairs for an infinite sequence of Vekua equations related to the original one. The necessity to count with an infinite number of exact solutions of different Vekua equations resulted in being an important obstacle for efficient construction of Taylor series for pseudoanalytic functions.

Nevertheless, as we show in this work, the Bers recursion formulae seem to have been specially designed for obtaining Taylor-type series for solutions of two-dimensional stationary Schrödinger equations admitting particular solutions which enjoy a peculiar property called in this work Condition S. The class of such Schrödinger equations is really wide. We show that if the potential in the Schrödinger equation is spherically symmetric or it is a function of one Cartesian or parabolic or elliptic variable, the corresponding Schrödinger equation belongs to this class. Moreover, the above-mentioned cases are only a few examples. In general, the Schrödinger equation belonging to the class must not necessarily admit separation of variables.

The main result of this work is a relatively simple procedure which allows us to construct explicitly a locally complete system of solutions of the Schrödinger equation by one known particular solution. Here, the local completeness is understood in the sense that any solution of the Schrödinger equation can be approximated arbitrarily closely by a linear combination of functions from this system in a neighbourhood of any point of the domain of interest. The global completeness is an open question; nevertheless, the possibility of obtaining a sequence of exact solutions with such a special property as the local completeness can be useful in different applications including qualitative analysis of solutions and numerical solution of boundary value problems. The main result is based on a chain of observations, some of them representing independent interest.

First of all, we observe (subsection 3.1) that given a particular solution of the stationary two-dimensional Schrödinger equation the corresponding Schrödinger operator can be factorized just as in a one-dimensional situation. In the considered two-dimensional case, the factorizing terms are operators $\partial_{\bar{z}} + \frac{\partial_z f_0}{f_0} C$ and $\partial_z - \frac{\partial_z f_0}{f_0} C$, where f_0 is a particular solution of the Schrödinger equation

$$(-\Delta + \nu) f = 0 \tag{1}$$

and C is the complex conjugation operator. This observation gives us a simple relation between the Schrödinger equation and the Vekua equation

$$\left(\partial_{\bar{z}} + \frac{\partial_z f_0}{f_0} C\right) w = 0. \tag{2}$$

Every solution of one of these equations can be transformed into a solution of the other and vice versa.

Next, we show that solutions of this equation are closely related to solutions of another Vekua equation

$$\left(\partial_{\bar{z}} - \frac{\partial_{\bar{z}} f_0}{f_0} C\right) W = 0 \tag{3}$$

which we call the main Vekua equation. For this equation, we always have a generating pair (F, G) in explicit form and the (F, G)-derivative of W (the operation introduced by L Bers) is a solution of (2). The (F, G)-antiderivative of w is a solution of (3). Moreover, the real part of W is necessarily a solution of (1) and the imaginary part of W is a solution of another Schrödinger equation with the potential $\left(-\nu + 2\left(\frac{|\nabla f_0|}{f_0}\right)^2\right)$ which is precisely the potential which would expect to be obtained after the Darboux transformation (see, e.g., [7]). We obtain a transformation which allows us to construct the imaginary part of W by its real part and vice versa, obtaining in this way an analogue of the Darboux transformation for the two-dimensional Schrödinger equation. Here, we should say that this transformation is not yet a long-sought definitive solution of the problem of a multidimensional generalization of the one-dimensional Darboux transformation (see, e.g., [10]) because it is not clear how to include in our consideration the eigenvalues of the operator. Nevertheless, it is certain progress in generalizing the Darboux transformation and deserves more attention.

Let f_0 be a function of some variable $\rho : f_0 = f_0(\rho)$ such that the expression $\Delta\rho/|\nabla\rho|^2$ is a function of ρ. We denote it by $s(\rho) = \frac{\Delta\rho}{|\nabla\rho|^2}$ and say that f_0 satisfies Condition S. We show that under this condition for equation (3) the formal powers in the sense of Bers can be constructed explicitly. It should be noted that formal powers play a part analogous to that of powers of the independent variable z in classical analytic function theory. They give us analogues of Taylor series expansions for solutions of (3), and locally any solution of (3) can be approximated arbitrarily closely by a finite number of first members of its Taylor series.

As was explained above, the real parts of formal powers are solutions of the Schrödinger equation (1) and therefore, similarly to real parts of powers of z which are very important in theory of harmonic functions and are widely used for numerical solution of boundary value problems for the Laplace equation, the real parts of formal powers give us a locally complete system of solutions.

For the sake of simplicity, we consider the Schrödinger equation with a real-valued potential, and in the last section we explain how our results can be generalized to the case of a complex-valued potential.

2. Some definitions and results from pseudoanalytic function theory

This section is based on the results presented in [3, 4]. Let Ω be a domain in $\mathbf{R}^2$. Throughout the whole paper, we suppose that Ω is a simply connected domain.

2.1. Generating pairs, derivative and antiderivative

Definition 1. *A pair of complex functions F and G possessing in Ω partial derivatives with respect to the real variables x and y is said to be a generating pair if it satisfies the inequality*

$$\operatorname{Im}(\overline{F}G) > 0 \qquad in \quad \Omega.$$

Denote $\partial_{\overline{z}} = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}$ and $\partial_z = \frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}$ (usually these operators are introduced with the factor $1/2$; nevertheless here it is somewhat more convenient to consider them without it). The following expressions are known as characteristic coefficients of the pair (F, G)

$$a_{(F,G)} = -\frac{\overline{F}G_{\overline{z}} - F_{\overline{z}}\overline{G}}{F\overline{G} - \overline{F}G}, \qquad b_{(F,G)} = \frac{FG_{\overline{z}} - F_{\overline{z}}G}{F\overline{G} - \overline{F}G},$$

$$A_{(F,G)} = -\frac{\overline{F}G_z - F_z\overline{G}}{F\overline{G} - \overline{F}G}, \qquad B_{(F,G)} = \frac{FG_z - F_zG}{F\overline{G} - \overline{F}G},$$

where the subindex $\overline{z}$ or z means the application of $\partial_{\overline{z}}$ or ∂_z, respectively.

Every complex function W defined in a subdomain of Ω admits the unique representation $W = \phi F + \psi G$ where the functions ϕ and ψ are real valued. Sometimes it is convenient to associate with the function W the function $\omega = \phi + \mathrm{i}\psi$. The correspondence between W and ω is one-to-one.

The (F, G)-derivative $\dot{W} = \frac{\mathrm{d}_{(F,G)}W}{\mathrm{d}z}$ of a function W exists and has the form

$$\dot{W} = \phi_z F + \psi_z G = W_z - A_{(F,G)}W - B_{(F,G)}\overline{W} \tag{4}$$

if and only if

$$\phi_{\overline{z}} F + \psi_{\overline{z}} G = 0. \tag{5}$$

This last equation can be rewritten in the following form:

$$W_{\overline{z}} = a_{(F,G)}W + b_{(F,G)}\overline{W}$$

which we call the Vekua equation. Solutions of this equation are called (F, G)-pseudoanalytic functions. If W is (F, G)-pseudoanalytic, the associated function ω is called (F, G)-pseudoanalytic of second kind.

Remark 2. The functions F and G are (F, G)-pseudoanalytic, and $\dot{F} \equiv \dot{G} \equiv 0$.

Definition 3. *Let (F, G) and (F_1, G_1) be two generating pairs in Ω. (F_1, G_1) is called successor of (F, G) and (F, G) is called predecessor of (F_1, G_1) if*

$$a_{(F_1,G_1)} = a_{(F,G)} \qquad \textit{and} \qquad b_{(F_1,G_1)} = -B_{(F,G)}.$$

The importance of this definition becomes obvious from the following statement.

Theorem 4. *Let W be an (F, G)-pseudoanalytic function and let (F_1, G_1) be a successor of (F, G). Then $\dot{W}$ is an (F_1, G_1)-pseudoanalytic function.*

Definition 5. *Let (F, G) be a generating pair. Its adjoint generating pair $(F, G)^* = (F^*, G^*)$ is defined by the formulae*

$$F^* = -\frac{2\overline{F}}{F\overline{G} - \overline{F}G}, \qquad G^* = \frac{2\overline{G}}{F\overline{G} - \overline{F}G}.$$

The (F, G)-integral is defined as follows:

$$\int_\Gamma W \,\mathrm{d}_{(F,G)}z = \frac{1}{2}\left(F(z_1)\,\mathrm{Re}\int_\Gamma G^* W \,\mathrm{d}z + G(z_1)\,\mathrm{Re}\int_\Gamma F^* W \,\mathrm{d}z\right)$$

where Γ is a rectifiable curve leading from z_0 to z_1.

If $W = \phi F + \psi G$ is an (F, G)-pseudoanalytic function where ϕ and ψ are real-valued functions then

$$\int_{z_0}^{z} \dot{W} \,\mathrm{d}_{(F,G)}z = W(z) - \phi(z_0)F(z) - \psi(z_0)G(z), \tag{6}$$

and as $\dot{F} = \dot{G} = 0$, this integral is path independent and represents the (F, G)-antiderivative of $\dot{W}$.

2.2. Generating sequences and Taylor series in formal powers

Definition 6. *A sequence of generating pairs* $\{(F_m, G_m)\}$, $m = 0, \pm1, \pm2, \ldots$ *, is called a generating sequence if* (F_{m+1}, G_{m+1}) *is a successor of* (F_m, G_m)*. If* $(F_0, G_0) = (F, G)$*, we say that* (F, G) *is embedded in* $\{(F_m, G_m)\}$.

Theorem 7. *Let* (F, G) *be a generating pair in* Ω*. Let* Ω_1 *be a bounded domain,* $\overline{\Omega_1} \subset \Omega$*. Then* (F, G) *can be embedded in a generating sequence in* Ω_1.

Definition 8. *A generating sequence* $\{(F_m, G_m)\}$ *is said to have period* $\mu > 0$ *if* $(F_{m+\mu}, G_{m+\mu})$ *is equivalent to* (F_m, G_m)*, that is their characteristic coefficients coincide.*

Let W be an (F, G)-pseudoanalytic function. Using a generating sequence in which (F, G) is embedded we can define the higher derivatives of W by the recursion formula

$$W^{[0]} = W; \qquad W^{[m+1]} = \frac{\mathrm{d}_{(F_m,G_m)} W^{[m]}}{\mathrm{d}z}, \qquad m = 1, 2, \ldots.$$

Definition 9. *The formal power* $Z_m^{(0)}(a, z_0; z)$ *with centre at* $z_0 \in \Omega$*, coefficient* a *and exponent 0 is defined as the linear combination of the generators* F_m*,* G_m *with real constant coefficients* λ*,* μ *chosen so that* $\lambda F_m(z_0) + \mu G_m(z_0) = a$*. The formal powers with exponents* $n = 1, 2, \ldots$ *are defined by the recursion formula*

$$Z_m^{(n+1)}(a, z_0; z) = (n+1) \int_{z_0}^{z} Z_{m+1}^{(n)}(a, z_0; \zeta)\, \mathrm{d}_{(F_m,G_m)}\zeta. \tag{7}$$

This definition implies the following properties.

(1) $Z_m^{(n)}(a, z_0; z)$ is an (F_m, G_m)-pseudoanalytic function of z.
(2) If a' and a'' are real constants, then $Z_m^{(n)}(a'+\mathrm{i}a'', z_0; z) = a' Z_m^{(n)}(1, z_0; z) + a'' Z_m^{(n)}(\mathrm{i}, z_0; z)$.
(3) The formal powers satisfy the differential relations

$$\frac{\mathrm{d}_{(F_m,G_m)} Z_m^{(n)}(a, z_0; z)}{\mathrm{d}z} = n Z_{m+1}^{(n-1)}(a, z_0; z).$$

(4) The asymptotic formulae

$$Z_m^{(n)}(a, z_0; z) \sim a(z - z_0)^n, \qquad z \to z_0$$

hold.

Assume now that

$$W(z) = \sum_{n=0}^{\infty} Z^{(n)}(a_n, z_0; z) \tag{8}$$

where the absence of the subscript m means that all the formal powers correspond to the same generating pair (F, G), and the series converges uniformly in some neighbourhood of z_0. It can be shown that the uniform limit of pseudoanalytic functions is pseudoanalytic, and that a uniformly convergent series of (F, G)-pseudoanalytic functions can be (F, G)-differentiated term by term. Hence, the function W in (8) is (F, G)-pseudoanalytic and its rth derivative admits the expansion

$$W^{[r]}(z) = \sum_{n=r}^{\infty} n(n-1)\cdots(n-r+1) Z_r^{(n-r)}(a_n, z_0; z).$$

From this the Taylor formulae for the coefficients are obtained

$$a_n = \frac{W^{[n]}(z_0)}{n!}. \tag{9}$$

Definition 10. *Let $W(z)$ be a given (F, G)-pseudoanalytic function defined for small values of $|z - z_0|$. The series*

$$\sum_{n=0}^{\infty} Z^{(n)}(a_n, z_0; z) \tag{10}$$

with the coefficients given by (9) is called the Taylor series of W at z_0, formed with formal powers.

The Taylor series always represents the function asymptotically:

$$W(z) - \sum_{n=0}^{N} Z^{(n)}(a_n, z_0; z) = O(|z - z_0|^{N+1}), \qquad z \to z_0, \tag{11}$$

for all N. This implies (since a pseudoanalytic function cannot have a zero of arbitrarily high order without vanishing identically) that the sequence of derivatives $\{W^{[n]}(z_0)\}$ determines the function W uniquely.

If the series (10) converges uniformly in a neighbourhood of z_0, it converges to the function W.

Theorem 11. *The formal Taylor expansion (10) of a pseudoanalytic function in formal powers defined by a periodic generating sequence converges in some neighbourhood of the centre.*

We will use also the following observation from [1, p 140] .

Proposition 12. *Let b be a complex function such that b_z is real valued, and let $W = u + \mathrm{i}v$ be a solution of the equation*

$$W_{\overline{z}} = b\overline{W}.$$

Then u is a solution of the equation

$$\partial_{\overline{z}}\partial_z u - (b\overline{b} + b_z)u = 0 \tag{12}$$

and v is a solution of the equation

$$\partial_{\overline{z}}\partial_z v - (b\overline{b} - b_z)v = 0. \tag{13}$$

3. Relationship between generalized analytic functions and solutions of the Schrödinger equation

3.1. Factorization of the Schrödinger operator

It is well known that if f_0 is a nonvanishing particular solution of the one-dimensional stationary Schrödinger equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \nu(x)\right) f(x) = 0$$

then the Schrödinger operator can be factorized as follows:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \nu(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{f_0'}{f_0}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{f_0'}{f_0}\right).$$

We start with a generalization of this result onto a two-dimensional situation. Consider the equation

$$(-\Delta + \nu) f = 0 \tag{14}$$

in some domain $\Omega \subset \mathbf{R}^2$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, ν and f are real-valued functions. We assume that f is a twice continuously differentiable function.

By C we denote the complex conjugation operator.

Theorem 13. *Let f_0 be a nonvanishing in Ω particular solution of (14). Then for any real-valued continuously twice differentiable function φ the following equality holds:*

$$(\Delta - \nu)\varphi = \left(\partial_{\bar z} + \frac{\partial_z f_0}{f_0} C\right)\left(\partial_z - \frac{\partial_z f_0}{f_0}\right)\varphi. \tag{15}$$

Proof. Consider

$$\begin{aligned}\left(\partial_{\bar z} + \frac{\partial_z f_0}{f_0} C\right)\left(\partial_z - \frac{\partial_z f_0}{f_0}\right)\varphi &= \Delta\varphi - \frac{|\partial_z f_0|^2}{f_0^2}\varphi - \partial_{\bar z}\left(\frac{\partial_z f_0}{f_0}\right)\varphi \\ &= \Delta\varphi - \frac{\Delta f_0}{f_0}\varphi = (\Delta - \nu)\varphi. \qquad \square\end{aligned}$$

Remark 14. As φ in (15) is a real-valued function, we can add the conjugation operator in the second first-order operator on the right-hand side, and then (15) takes the form

$$(\Delta - \nu)\varphi = \left(\partial_{\bar z} + \frac{\partial_z f_0}{f_0} C\right)\left(\partial_z - \frac{\partial_z f_0}{f_0} C\right)\varphi.$$

The operator $\partial_z - \frac{\partial_z f_0}{f_0} I$, where I is the identity operator, can be represented in the form

$$\partial_z - \frac{\partial_z f_0}{f_0} I = f_0 \partial_z f_0^{-1} I.$$

Let us introduce the following notation $P = f_0 \partial_z f_0^{-1} I$. Due to theorem 13, if f_0 is a nonvanishing solution of (14), the operator P transforms solutions of (14) into solutions of the equation

$$\left(\partial_{\bar z} + \frac{\partial_z f_0}{f_0} C\right) w = 0. \tag{16}$$

Note that the operator ∂_z applied to a real-valued function φ can be regarded as a kind of gradient, and if we know that $\partial_z \varphi = \Phi$ in a whole complex plane or in a convex domain, where $\Phi = \Phi_1 + \mathrm{i}\Phi_2$ is a given complex-valued function such that its real part Φ_1 and imaginary part Φ_2 satisfy the equation

$$\partial_y \Phi_1 + \partial_x \Phi_2 = 0, \tag{17}$$

then we can reconstruct φ up to an arbitrary real constant c in the following way:

$$\varphi(x, y) = \int_{x_0}^{x} \Phi_1(\eta, y)\, \mathrm{d}\eta - \int_{y_0}^{y} \Phi_2(x_0, \xi)\, \mathrm{d}\xi + c \tag{18}$$

where (x_0, y_0) is an arbitrary fixed point in the domain of interest.

By A we denote the integral operator in (18):

$$A[\Phi](x, y) = \int_{x_0}^{x} \Phi_1(\eta, y)\, \mathrm{d}\eta - \int_{y_0}^{y} \Phi_2(x_0, \xi)\, \mathrm{d}\xi + c.$$

Note that formula (18) can be easily extended to any simply connected domain by considering the integral along an arbitrary rectifiable curve Γ leading from (x_0, y_0) to (x, y)

$$\varphi(x, y) = \int_\Gamma \Phi_1 \, dx - \Phi_2 \, dy + c.$$

Thus, if Φ satisfies (17), there exists a family of real-valued functions φ such that $\partial_z \varphi = \Phi$, given by the formula $\varphi = A[\Phi]$.

In a similar way, we define the operator $\overline{A}$ corresponding to $\partial_{\overline{z}}$:

$$\overline{A}[\Phi](x, y) = \int_{x_0}^{x} \Phi_1(\eta, y) \, d\eta + \int_{y_0}^{y} \Phi_2(x_0, \xi) \, d\xi + c.$$

Consider the operator $S = f_0 A f_0^{-1} I$. It is clear that $PS = I$.

Proposition 15 ([6])**.** *Let f_0 be a nonvanishing particular solution of (14) and w be a solution of (16). Then the function $f = Sw$ is a solution of (14).*

Proposition 16 ([6])**.** *Let f be a solution of (14). Then,*

$$SPf = f + cf_0$$

where c is an arbitrary real constant.

Theorem 13 together with proposition 15 show us that equation (14) is equivalent to the Vekua equation (16) in the following sense. Every solution of one of these equations can be transformed into a solution of the other equation and vice versa.

3.2. The main Vekua equation

Equation (16) is closely related to the following Vekua equation:

$$\left(\partial_{\overline{z}} - \frac{\partial_{\overline{z}} f_0}{f_0} C\right) W = 0. \tag{19}$$

To see this let us observe that the pair of functions

$$F = f_0 \qquad \text{and} \qquad G = \frac{\mathrm{i}}{f_0} \tag{20}$$

is a generating pair for (19). Then the corresponding characteristic coefficients $A_{(F,G)}$ and $B_{(F,G)}$ have the form

$$A_{(F,G)} = 0, \qquad B_{(F,G)} = \frac{\partial_z f_0}{f_0},$$

and the (F, G)-derivative according to (4) is defined as follows:

$$\dot{W} = W_z - \frac{\partial_z f_0}{f_0} \overline{W} = \left(\partial_z - \frac{\partial_z f_0}{f_0} C\right) W.$$

Comparing $B_{(F,G)}$ with the coefficient in (16) and due to theorem 4 we obtain the following statement.

Proposition 17. *Let W be a solution of (19). Then its (F, G)-derivative, the function $w = \dot{W}$ is a solution of (16).*

This result can be verified also by a direct substitution.

According to (6) and taking into account that

$$F^* = -\mathrm{i} f_0 \qquad \text{and} \qquad G^* = 1/f_0,$$

the (F, G)-antiderivative has the form

$$\int_{z_0}^{z} w(\zeta)\,\mathrm{d}_{(F,G)}\zeta = \frac{1}{2}\left(f_0(z)\,\mathrm{Re}\int_{z_0}^{z}\frac{w(\zeta)}{f_0(\zeta)}\,\mathrm{d}\zeta - \frac{\mathrm{i}}{f_0(z)}\,\mathrm{Re}\int_{z_0}^{z}\mathrm{i}f_0(\zeta)w(\zeta)\,\mathrm{d}\zeta\right)$$
$$= \frac{1}{2}\left(f_0(z)\,\mathrm{Re}\int_{z_0}^{z}\frac{w(\zeta)}{f_0(\zeta)}\,\mathrm{d}\zeta + \frac{\mathrm{i}}{f_0(z)}\,\mathrm{Im}\int_{z_0}^{z}f_0(\zeta)w(\zeta)\,\mathrm{d}\zeta\right), \tag{21}$$

and we obtain the following statement.

Proposition 18. *Let w be a solution of (16). Then the function*

$$W(z) = \int_{z_0}^{z} w(\zeta)\,\mathrm{d}_{(F,G)}\zeta$$

is a solution of (19).

Let ϕ and ψ be real-valued functions. It is easy to see that the function $W = \phi f_0 + \mathrm{i}\psi/f_0$ is a solution of (19) if and only if ϕ and ψ satisfy the equation $\psi_{\bar z} - \mathrm{i}f_0^2\phi_{\bar z} = 0$ which is equivalent to the system

$$\psi_x + f_0^2\phi_y = 0, \qquad \psi_y - f_0^2\phi_x = 0$$

defining so-called p-analytic functions (see [9, 6]) with $p = f_0^2$.

Proposition 19. *Let W be a solution of (19). Then $u = \mathrm{Re}\,W$ is a solution of (14) and $v = \mathrm{Im}\,W$ is a solution of the equation*

$$\left(\Delta + \nu - 2\left(\frac{|\nabla f_0|}{f_0}\right)^2\right)v = 0. \tag{22}$$

Proof. Observe that the coefficient $b = \frac{\partial_{\bar z} f_0}{f_0}$ in (19) satisfies the condition of proposition 12:

$$b_z = \frac{\Delta f_0}{f_0} - \left(\frac{|\partial_{\bar z} f_0|}{f_0}\right)^2 = \nu - \left(\frac{|\partial_{\bar z} f_0|}{f_0}\right)^2.$$

Thus, according to proposition 12, u is a solution of (12) and v is a solution of (13). Calculating the expressions $b\overline{b} + b_z = \nu$ and $b\overline{b} - b_z = 2\left(\frac{|\nabla f_0|}{f_0}\right)^2 - \nu$ we complete the proof. □

Proposition 20. *Let u be a solution of (14). Then the function*

$$v \in \ker\left(\Delta + \nu - 2\left(\frac{|\nabla f_0|}{f_0}\right)^2\right)$$

such that $W = u + \mathrm{i}v$ is a solution of (19), is constructed according to the formula

$$v = f_0^{-1}\overline{A}\left(\mathrm{i}f_0^2\partial_{\bar z}\left(f_0^{-1}u\right)\right). \tag{23}$$

It is unique up to an additive term cf_0^{-1} where c is an arbitrary real constant.

Given $v \in \ker\left(\Delta + \nu - 2\left(\frac{|\nabla f_0|}{f_0}\right)^2\right)$, the corresponding $u \in \ker(\Delta - \nu)$ can be constructed as follows:

$$u = -f_0\overline{A}\left(\mathrm{i}f_0^{-2}\partial_{\bar z}(f_0 v)\right) \tag{24}$$

up to an additive term cf_0.

Proof. Consider equation (19). Let $W = \phi f_0 + \mathrm{i}\psi/f_0$ be its solution. Then the equation

$$\psi_{\bar z} - \mathrm{i}f_0^2\phi_{\bar z} = 0 \tag{25}$$

is valid. Note that if $u = \operatorname{Re} W$ then $\phi = u/f_0$. Given ϕ, ψ is easily found from (25):

$$\psi = \overline{A}\big(\mathrm{i} f_0^2 \phi_{\overline{z}}\big).$$

It can be verified that the expression $\overline{A}\big(\mathrm{i} f_0^2 \phi_{\overline{z}}\big)$ makes sense, that is $\partial_x\big(f_0^2\phi_x\big) + \partial_y\big(f_0^2\phi_y\big) = 0$.

By proposition 19, the function $v = f_0^{-1}\psi$ is a solution of (22). Thus, we obtain (23). Let us note that as the operator $\overline{A}$ reconstructs the scalar function up to an arbitrary real constant, the function v in formula (23) is uniquely determined up to an additive term cf_0^{-1} where c is an arbitrary real constant.

Equation (24) is proved in a similar way. □

Remark 21. The potential in the Schrödinger equation (22) has the form of a potential obtained after the Darboux transformation (cf [7, 8]) and thus formulae (23) and (24) can be considered as a two-dimensional analogue of the Darboux transformation, though it is not clear how to include in our consideration the eigenvalues of the operator.

Remark 22. When $\nu \equiv 0$ and $f_0 \equiv 1$, equalities (23) and (24) turn into the well known in complex analysis formulae for constructing conjugate harmonic functions.

Remark 23. Equation (19) can be written as follows:

$$\big(f_0\partial_{\overline{z}} f_0^{-1} P^+ + \mathrm{i} f_0^{-1}\partial_{\overline{z}} f_0 P^-\big)\, W = 0 \tag{26}$$

where $P^+ = \frac{1}{2}(I + C)$ and $P^- = \frac{1}{2\mathrm{i}}(I - C)$.

The form of the operator in (26) suggests the following form of an inverse operator:

$$H\Phi = \tfrac{1}{2}\left(f_0\overline{A}\left(f_0^{-1}\Phi\right) + \mathrm{i} f_0^{-1}\overline{A}(\mathrm{i} f_0\Phi)\right) \tag{27}$$

where Φ must be such function that the expressions $\overline{A}\big(f_0^{-1}\Phi\big)$ and $\overline{A}(\mathrm{i} f_0\Phi)$ make sense.

Proposition 24. *The function $W = H\Phi$ defined by (27) is a solution of (26) and equivalently of (19) if and only if $\overline{\Phi}$ is a solution of (16).*

Proof. Assume that Φ is such that the expressions $\overline{A}\big(f_0^{-1}\Phi\big)$ and $\overline{A}(\mathrm{i} f_0\Phi)$ make sense, that is

$$\partial_y \operatorname{Re}\big(f_0^{-1}\Phi\big) - \partial_x \operatorname{Im}\big(f_0^{-1}\Phi\big) = 0 \tag{28}$$

and

$$\partial_y \operatorname{Re}(\mathrm{i} f_0\Phi) - \partial_x \operatorname{Im}(\mathrm{i} f_0\Phi) = 0. \tag{29}$$

In this case, let us substitute the function $W = H\Phi$ in (26). We have

$$\begin{aligned}
&\tfrac{1}{2}\left(f_0\partial_{\overline{z}} f_0^{-1} P^+ + \mathrm{i} f_0^{-1}\partial_{\overline{z}} f_0 P^-\right)\left(f_0\overline{A}\left(f_0^{-1}\Phi\right) + \mathrm{i} f_0^{-1}\overline{A}\,(\mathrm{i} f_0\Phi)\right)\\
&\qquad = \tfrac{1}{2}\left(f_0\partial_{\overline{z}}\overline{A}\left(f_0^{-1}\Phi\right) + \mathrm{i} f_0^{-1}\partial_{\overline{z}}\overline{A}\,(\mathrm{i} f_0\Phi)\right) = \tfrac{1}{2}(\Phi - \Phi) = 0.
\end{aligned}$$

Now let us prove that (28) and (29) are equivalent to the fact that $\overline{\Phi}$ is a solution of (16). Denote $\phi = \operatorname{Re}\Phi$ and $\psi = \operatorname{Im}\Phi$. Then (28) and (29) can be written as follows:

$$\partial_x\left(\frac{\psi}{f_0}\right) - \partial_y\left(\frac{\phi}{f_0}\right) = 0$$

and

$$\partial_x(f_0\phi) + \partial_y(f_0\psi) = 0.$$

The last two equalities are equivalent to the system

$$\partial_x\phi + \partial_y\psi = -\frac{\partial_x f_0}{f_0}\phi - \frac{\partial_y f_0}{f_0}\psi,$$

$$\partial_x \psi - \partial_y \phi = \frac{\partial_x f_0}{f_0}\psi - \frac{\partial_y f_0}{f_0}\phi,$$

which can be rewritten as the equation

$$\partial_{\bar z}\overline{\Phi} = -\frac{\partial_z f_0}{f_0}\Phi. \qquad \square$$

Remark. This proposition shows us that in the case of equation (19) the (F, G)-antiderivative can be calculated using (27). Indeed, let W be a solution of (19). Consider its (F, G)-derivative $\dot W$ which is a solution of (16) due to proposition 17. It can be written as follows:

$$\dot W = f_0 \partial_z \left(f_0^{-1} u\right) + \mathrm{i} f_0^{-1} \partial_z (f_0 v),$$

where $u = \operatorname{Re} W$ and $v = \operatorname{Im} W$. Consider

$$\begin{aligned} HC\dot W &= \tfrac{1}{2}\left(f_0 \overline{A}\left(f_0^{-1} C \dot W\right) + \mathrm{i} f_0^{-1} \overline{A}\left(\mathrm{i} f_0 C \dot W\right)\right) \\ &= \tfrac{1}{2} f_0 \overline{A}\left(\partial_{\bar z}\left(f_0^{-1} u\right)\right) - f_0 \overline{A}\left(\mathrm{i} f_0^{-2} \partial_{\bar z}(f_0 v)\right) \\ &\quad + \mathrm{i} f_0^{-1} \overline{A}\left(\mathrm{i} f_0^2 \partial_{\bar z}\left(f_0^{-1} u\right)\right) + \mathrm{i} f_0^{-1} \overline{A}\left(\partial_{\bar z}(f_0 v)\right) \\ &= \tfrac{1}{2}\left(u + \mathrm{i} v - f_0 \overline{A}\left(\mathrm{i} f_0^{-2} \partial_{\bar z}(f_0 v)\right) + \mathrm{i} f_0^{-1} \overline{A}\left(\mathrm{i} f_0^2 \partial_{\bar z}\left(f_0^{-1} u\right)\right)\right). \end{aligned}$$

From here, due to proposition 19 we obtain

$$HC\dot W = u + \mathrm{i} v + c_1 f_0 + \frac{\mathrm{i} c_2}{f_0},$$

where c_1 and c_2 are arbitrary real constants.

Thus, application of the operator HC to solutions of (16) gives us exactly the same result as the (F, G)-antiderivative defined by (21).

4. Taylor series in formal powers for pseudoanalytic functions and solutions of the Schrödinger equation

In this section, we show how for a quite ample class of potentials by one known particular solution of (14) one can always construct an infinite sequence of its solutions possessing the property of local completeness.

4.1. Condition S

Lemma 26. *Let φ be a nonvanishing analytic function. Then solutions of the equations*

$$W_{\bar z} = b\overline{W} \tag{30}$$

and

$$w_{\bar z} = \frac{\varphi}{\overline{\varphi}} b\overline{w} \tag{31}$$

are related in the following way. If W is a solution of (30) then $w = \varphi W$ is a solution of (31), and if w is a solution of (31) then $W = w/\varphi$ is a solution of (30).

The proof of this statement is obvious.

Proposition 27. *Let f_0 be a function of some real variable ρ: $f_0 = f_0(\rho)$ such that for some real-valued nonvanishing function η the equation*

$$\partial_{\bar z}(\eta \partial_z \rho) = 0 \qquad \text{in} \quad \Omega \tag{32}$$

holds. Denote $\varphi = \mathrm{i}\eta\rho_z$. *Then,*

$$\frac{\partial_z f_0}{f_0} = -\frac{\varphi}{\overline{\varphi}} \frac{\partial_{\overline{z}} f_0}{f_0},$$

and if W *is a solution of (19), the function* $w = \varphi W$ *is a solution of (16) and vice versa, if* w *is a solution of (16), the function* $W = w/\varphi$ *is a solution of (19).*

Proof. Consider the expression $\frac{\partial_z f_0}{f_0} = \frac{f_0' \rho_z}{f_0}$. Observe that $\overline{\varphi}\rho_z = -\varphi\rho_{\overline{z}}$. Then,

$$\frac{\partial_z f_0}{f_0} = -\frac{\varphi}{\overline{\varphi}} \frac{f_0' \rho_{\overline{z}}}{f_0} = -\frac{\varphi}{\overline{\varphi}} \frac{\partial_{\overline{z}} f_0}{f_0}.$$

From (32) it is evident that φ is analytic. Then denoting $b = \frac{\partial_{\overline{z}} f_0}{f_0}$ we see that (19) is equation (30) from lemma 26 and equation (16) is equation (31). Thus, by lemma 26 we obtain the result. □

Consider the following condition introduced in [5].

Condition 28. *(Condition S). Let* f_0 *be a function of some variable* $\rho : f_0 = f_0(\rho)$ *such that* $\frac{\Delta\rho}{|\nabla\rho|^2}$ *is a function of* ρ. *We denote it by* $s(\rho) = \frac{\Delta\rho}{|\nabla\rho|^2}$.

The following proposition gives us a description of all possible solutions of (32).

Proposition 29. *For a real-valued nontrivial function* ρ *there exists a real-valued nonvanishing function* η *such that (32) holds if and only if* ρ *satisfies condition 28.*

Proof. Let ρ satisfy condition 28. Denote $\eta = \mathrm{e}^{-S}$ where S is the antiderivative of s with respect to ρ. Consider

$$\partial_{\overline{z}}(\eta\partial_z\rho) = \partial_{\overline{z}}(\mathrm{e}^{-S}\rho_z) = -s\,\mathrm{e}^{-S}|\nabla\rho|^2 + \mathrm{e}^{-S}\Delta\rho = 0.$$

Now assume that for ρ there exists a real-valued nonvanishing function η such that (32) holds. Then $\Delta\rho + \frac{\eta_{\overline{z}}}{\eta}\rho_z = 0$ or in another form

$$\frac{\eta_{\overline{z}}}{\eta} = -\frac{\Delta\rho}{|\nabla\rho|^2}\rho_{\overline{z}}.$$

If ρ is harmonic then condition 28 is obviously fulfilled, so let us consider the opposite case assuming that ρ is not harmonic. The last equation can be written as follows:

$$\nabla \ln \eta = \mu\nabla\rho,$$

where $\mu = -\frac{\Delta\rho}{|\nabla\rho|^2}$. In order that the product $\mu\nabla\rho$ be a gradient it is necessary that $[\nabla\mu \times \nabla\rho] = 0$ which implies that $\mu = \mu(\rho)$. □

Thus, due to proposition 29 we have that the function φ from proposition 27 has the form $\varphi = \mathrm{i}\,\mathrm{e}^{-S}\rho_z$ where $S(\rho) = \int \frac{\Delta\rho}{|\nabla\rho|^2}\,\mathrm{d}\rho$.

4.2. Some examples of functions satisfying Condition S

Examples of variables ρ which satisfy Condition S are numerous and important in applications. As was mentioned above any harmonic function ρ fulfils the condition and obviously $\eta \equiv 1$ in (32). That is, for example, $\rho(x, y) = a_1 x + a_2 y$, where a_1 and a_2 are arbitrary real constants, or $\rho(x, y) = xy$ belong to that class.

An important example is $\rho(x, y) = r = \sqrt{x^2 + y^2}$. In this case, $\frac{\Delta\rho}{|\nabla\rho|^2} = \frac{1}{\rho}$. The parabolic coordinate $\rho(x, y) = r + x$ also fulfils Condition S: $\frac{\Delta\rho}{|\nabla\rho|^2} = \frac{1}{2\rho}$.

Consider the elliptic coordinates μ and θ:

$$x = \frac{a}{2}\cosh\mu\cos\theta, \qquad y = \frac{a}{2}\sinh\mu\sin\theta.$$

It is convenient to consider the magnitudes

$$r_1 = \sqrt{\left(x + \frac{a}{2}\right)^2 + y^2} = \frac{a}{2}(\cosh\mu + \cos\theta),$$

$$r_2 = \sqrt{\left(x - \frac{a}{2}\right)^2 + y^2} = \frac{a}{2}(\cosh\mu - \cos\theta).$$

Let us verify Condition S for instance for the variable μ. It is somewhat easier to consider $\rho = a\cosh\mu$. If Condition S is fulfilled for ρ then it is obviously true for μ. We have $\rho = r_1 + r_2$, and

$$\rho_z = \frac{\overline{z} + a/2}{r_1} + \frac{\overline{z} - a/2}{r_2}.$$

Then,

$$|\nabla\rho|^2 = \left(\frac{\overline{z} + a/2}{r_1} + \frac{\overline{z} - a/2}{r_2}\right)\left(\frac{z + a/2}{r_1} + \frac{z - a/2}{r_2}\right)$$
$$= \frac{\rho^2 - a^2}{r_1 r_2}$$

and

$$\Delta\rho = \frac{\rho}{r_1 r_2}.$$

We obtain

$$\frac{\Delta\rho}{|\nabla\rho|^2} = \frac{\rho}{\rho^2 - a^2}.$$

Thus, in all considered cases when ρ is one of the Cartesian coordinates, when $\rho = r$, when ρ is one of the parabolic coordinates or when ρ is one of the elliptic coordinates, Condition S is fulfilled. Moreover, as the Laplacian admits separation of variables in all the mentioned coordinate systems, then if the potential ν is a function of such ρ, there exists a particular solution of (14) $f_0 = f_0(\rho)$, and the results of this section are applicable to all Schrödinger equations with potentials depending on such ρ.

We should emphasize first that these are only some examples which definitely do not exhaust all interesting in applications situations that can be covered by Condition S. And second, in order that such a solution $f_0 = f_0(\rho)$ exists fulfilling Condition S, it is obviously not necessary that ν be a function of ρ.

4.3. Explicitly constructed generating sequence for the main Vekua equation with f_0 satisfying Condition S

In what follows, we assume that f_0 is a nonvanishing solution of (14) satisfying Condition S.

Theorem 30. *Let $\varphi = \mathrm{i}\,\mathrm{e}^{-S}\rho_z \neq 0$ in Ω. Then the generating pair (F, G) with $F = f_0$ and $G = \mathrm{i}/f_0$ is embedded in the generating sequence (F_m, G_m), $m = 0, \pm1, \pm2, \ldots$ with $F_m = \varphi^m F$ and $G_m = \varphi^m G$.*

Proof. First of all let us show that (F_m, G_m) is a generating pair for $m = \pm1, \pm2, \ldots$. Indeed, we have

$$\operatorname{Im}(\overline{F}_m G_m) = \operatorname{Im}(\overline{\varphi}^m \overline{F}\varphi^m G) = \operatorname{Im}(|\varphi|^{2m}\overline{F}G) > 0.$$

Taking into account that $\varphi_{\bar z} = 0$ it is easy to obtain the following equalities:

$$a_{(F_m,G_m)} = |\varphi|^{2m} a_{(F,G)} \equiv 0, \qquad b_{(F_m,G_m)} = \frac{\varphi^m}{\overline{\varphi}^m} b_{(F,G)}$$

and

$$B_{(F_{m-1},G_{m-1})} = \frac{\varphi^{m-1}}{\overline{\varphi}^{m-1}} B_{(F,G)}.$$

We should verify the equality

$$b_{(F_m,G_m)} = -B_{(F_{m-1},G_{m-1})} \tag{33}$$

which turns into the equality

$$\frac{\varphi}{\overline{\varphi}} b_{(F,G)} = -B_{(F,G)}.$$

As $b_{(F,G)} = \frac{\partial_{\bar z} f_0}{f_0}$ and $B_{(F,G)} = \frac{\partial_z f_0}{f_0}$ by proposition 27 we obtain that (33) is true. Thus, the sequence (F_m, G_m), $m = 0, \pm 1, \pm 2, \ldots$ satisfies the conditions of definition 6 and therefore it is a generating sequence. □

This theorem opens the way for explicit construction of formal powers of any order $n \geqslant 0$ corresponding to the generating pair $(f_0, \mathrm{i}/f_0)$ as well as to any generating pair embedded in the sequence proposed in theorem 30. As a consequence, Bers' theory of series expansion for pseudoanalytic functions can be used in order to obtain explicitly Taylor series in formal powers for solutions of the Schrödinger equation (14) due to propositions 19 and 20.

Due to (11) we have that any pseudoanalytic function W can be approximated at any point $z_0 \in \Omega$ with an arbitrary precision by first N members of its Taylor series in formal powers. As any solution of (14) is a real part of some pseudoanalytic function W satisfying (19), it can also be approximated with arbitrary precision by the sum of real parts of the first N members of the Taylor series in formal powers of the function W.

Definition 31. *Let $u(z)$ be a given solution of (14) defined for small values of $|z - z_0|$, and let $W(z)$ be a solution of (19) constructed according to proposition 20 such that* Re $W = u$*. The series*

$$\sum_{n=0}^{\infty} \operatorname{Re} Z^{(n)}(a_n, z_0; z) \tag{34}$$

with the coefficients given by (9) is called the Taylor series of u at z_0, formed with formal powers.

Theorem 32.

$$u(z) - \sum_{n=0}^{N} \operatorname{Re} Z^{(n)}(a_n, z_0; z) = O(|z - z_0|^{N+1}), \qquad z \to z_0, \tag{35}$$

for all N, and if the series (34) converges uniformly in a neighbourhood of z_0, it converges to the function u.

Proof. This is a direct consequence of (11). □

Due to theorem 30 we are able to construct $Z^{(n)}(a_n, z_0; z)$ explicitly in many practically interesting cases. The simplest case is when f_0 is a function of one Cartesian variable:

$f_0 = f_0(y)$, that is we consider the equation

$$-\Delta f(x, y) + \nu(y) f(x, y) = 0 \qquad \text{in} \quad \Omega. \tag{36}$$

Let us make a useful observation. In this case $\rho_z = -\mathrm{i}$ and $\varphi = 1$. Thus, $(F_m, G_m) = (F, G), m = 0, \pm 1, \pm 2, \ldots,$ and obviously we have a periodic generating sequence with the period 1. Consequently, according to theorem 11 in the case under consideration we can guarantee not only the approximations (11) and (35) but also the convergence of the Taylor series in formal powers in some neighbourhood of the centre. Let us consider the following example.

Example 33. Let Ω be the unit circle with centre at the origin. Consider equation (36) with $\nu(y) = 6/(y+1)^2$. One particular solution depending on y only can be chosen as follows $f_0(y) = (y+1)^3$. It is easy to find another solution of (36). We choose it in the following form $u(y) = (y+1)^{-2}$. Next, we construct the function v such that $W = u + \mathrm{i}v$ be a solution of (19). Using (23) we obtain $v = 5x(y+1)^{-3} + c(y+1)^{-3}$ where c is an arbitrary real constant. We choose it equal to 0, so $v = 5x(y+1)^{-3}$. It can be easily verified that the function $W = (y+1)^{-2} + 5\mathrm{i}x(y+1)^{-3}$ is indeed a solution of (19) where $\frac{\partial_{\bar{z}} f_0}{f_0} = 3\mathrm{i}(y+1)^{-1}$. Our aim is to find the Taylor series in formal powers of the function W at the origin.

We find that $W(0) = 1$. Then using the definition of formal powers, we find $Z^{(0)}(1, 0; z) = (y+1)^3$. In order to construct $Z^{(1)}$, we need to calculate $\dot{W}(0)$. We have $\dot{W}(z) = \partial_z W(z) + 3\mathrm{i}(y+1)^{-1}\overline{W}(z) = 10\mathrm{i}(y+1)^{-3}$. Thus, $\dot{W}(0) = 10\mathrm{i}$. In order to apply (7), we find $Z^{(0)}(10\mathrm{i}, 0; z) = 10\mathrm{i}/(y+1)^3$. Then using (7) and remark 25 we have

$$Z^{(1)}(10\mathrm{i}, 0; z) = HCZ^{(0)}(10\mathrm{i}, 0; z) = (y+1)^{-2} - (y+1)^3 + 5\mathrm{i}x(y+1)^{-3}.$$

The function $\widetilde{W}(z) = Z^{(0)}(1, 0; z) + Z^{(1)}(10\mathrm{i}, 0; z)$ should satisfy the equality $W(z) - \widetilde{W}(z) = O(|z|^2)$ when $z \to 0$. We see that in fact $\widetilde{W}$ coincides with W as could be expected due to the fact that $\dot{W}$ is the generating function G multiplied by a real constant and hence $W^{[2]} \equiv 0$.

Formal powers' Property 2 from subsection 2.2 together with theorem 30 allows us to construct in explicit form a locally complete system of solutions of (19) and of (14) in the following sense. Due to theorem 30 when f_0 fulfils Condition S, we are able to construct the formal powers $Z^{(n)}(1, z_0; z)$ and $Z^{(n)}(\mathrm{i}, z_0; z), n = 0, 1, 2, \ldots$ corresponding to any point $z_0 \in \Omega$. Due to Property 2 of formal powers, we have that $Z^{(n)}(a, z_0; z)$ for any Taylor coefficient a can be easily expressed through $Z^{(n)}(1, z_0; z)$ and $Z^{(n)}(\mathrm{i}, z_0; z)$. Thus, for any solution W of (19) there exists a linear combination of $Z^{(n)}(1, z_0; z)$ and $Z^{(n)}(\mathrm{i}, z_0; z), n = 0, 1, 2, \ldots, N$ such that (11) is valid.

Hence, for any solution u of (14) there exists a linear combination of $\operatorname{Re} Z^{(n)}(1, z_0; z)$ and $\operatorname{Re} Z^{(n)}(\mathrm{i}, z_0; z), n = 0, 1, 2, \ldots, N$ such that (35) holds.

Let us illustrate the procedure of construction of this locally complete system on the following example.

Example 34. Let Ω be the unit circle with centre at the origin and α, β positive real constants greater than 1. For the Schrödinger equation (14) with

$$\nu(x, y) = -\frac{1}{4}\left(\frac{1}{(x+\alpha)^2} + \frac{1}{(y+\beta)^2}\right) \tag{37}$$

we have the particular solution $f_0(x, y) = \sqrt{(x+\alpha)(y+\beta)}$. Denote $\rho = (x+\alpha)(y+\beta)$. Then Condition S is fulfilled and we have that the function φ from theorem 30 is defined as follows $\varphi = \mathrm{i}\,\mathrm{e}^{-S}\rho_z = z + c$, where $c = \alpha + \mathrm{i}\beta$. Let us construct the first formal powers

$Z^{(n)}(1, 0; z)$ and $Z^{(n)}(\mathrm{i}, 0; z)$. By definition 9, we have

$$Z^{(0)}(1, 0; z) = \sqrt{\frac{(x+\alpha)(y+\beta)}{\alpha\beta}} \qquad \text{and} \qquad Z^{(0)}(\mathrm{i}, 0; z) = \mathrm{i}\sqrt{\frac{\alpha\beta}{(x+\alpha)(y+\beta)}}.$$

In order to construct $Z^{(1)}(1, 0; z)$ and $Z^{(1)}(\mathrm{i}, 0; z)$ by formula (7) we need first $Z_1^{(0)}(1, 0; z)$ and $Z_1^{(0)}(\mathrm{i}, 0; z)$. We have that $F_1 = (z+c)\sqrt{(x+\alpha)(y+\beta)}$ and $G_1 = \mathrm{i}(z+c)/\sqrt{(x+\alpha)(y+\beta)}$. Then $Z_1^{(0)}(1, 0; z) = \lambda_1 F_1(z) + \mu_1 G_1(z)$ with λ_1 and μ_1 satisfying the equality $\lambda_1 F_1(0) + \mu_1 G_1(0) = 1$ and $Z_1^{(0)}(\mathrm{i}, 0; z) = \lambda_2 F_1(z) + \mu_2 G_1(z)$ with λ_2 and μ_2 satisfying the equality $\lambda_2 F_1(0) + \mu_2 G_1(0) = \mathrm{i}$. We obtain that

$$\lambda_1 = \frac{\sqrt{\alpha}}{\sqrt{\beta}(\alpha^2+\beta^2)}, \qquad \mu_1 = -\frac{\sqrt{\alpha\beta^3}}{(\alpha^2+\beta^2)} \tag{38}$$

and

$$\lambda_2 = \frac{\sqrt{\beta}}{\sqrt{\alpha}(\alpha^2+\beta^2)}, \qquad \mu_2 = \frac{\sqrt{\alpha^3\beta}}{(\alpha^2+\beta^2)}. \tag{39}$$

Consider

$$\begin{aligned} Z^{(1)}(a, 0; z) &= \int_0^z Z_1^{(0)}(a, 0; \zeta)\, \mathrm{d}_{(F,G)}\zeta \\ &= \frac{1}{2}\Big(\sqrt{(x+\alpha)(y+\beta)}\,\mathrm{Re}\int_0^z \left(\lambda(\zeta+c) + \mu\frac{(\zeta+c)\mathrm{i}}{(x'+\alpha)(y'+\beta)}\right)\mathrm{d}\zeta \\ &\quad + \frac{\mathrm{i}}{\sqrt{(x+\alpha)(y+\beta)}}\,\mathrm{Im}\int_0^z (\lambda(\zeta+c)(x'+\alpha)(y'+\beta) + \mu(\zeta+c)\mathrm{i})\,\mathrm{d}\zeta\Big) \end{aligned}$$

where λ and μ are real numbers such that $\lambda F_1(0) + \mu G_1(0) = a$ and $\zeta = x' + \mathrm{i}y'$. We have

$$\begin{aligned} \mathrm{Re}\int_0^z (\zeta+c)\,\mathrm{d}\zeta &= \int_0^1 ((x^2-y^2)t + \alpha x - \beta y)\,\mathrm{d}t = \frac{(x^2-y^2)}{2} + \alpha x - \beta y, \\ \mathrm{Re}\int_0^z \frac{(\zeta+c)\mathrm{i}\,\mathrm{d}\zeta}{(x'+\alpha)(y'+\beta)} &= -\int_0^1 \frac{2xyt + \alpha y + \beta x}{(xt+\alpha)(yt+\beta)}\,\mathrm{d}t \\ &= -2xy\left(\frac{\alpha}{x(\alpha y - \beta x)}\ln\left(\frac{x+\alpha}{\alpha}\right) - \frac{\beta}{y(\alpha y - \beta x)}\ln\left(\frac{y+\beta}{\beta}\right)\right) \\ &\quad - \frac{\alpha y + \beta x}{\alpha y - \beta x}\ln\frac{\alpha(y+\beta)}{\beta(x+\alpha)} \\ &= \ln\left(\frac{\alpha\beta}{(x+\alpha)(y+\beta)}\right), \end{aligned}$$

and

$$\begin{aligned} \mathrm{Im}\int_0^z (\zeta+c)\mathrm{i}\,\mathrm{d}\zeta &= \mathrm{Re}\int_0^z (\zeta+c)\,\mathrm{d}\zeta = \frac{(x^2-y^2)}{2} + \alpha x - \beta y, \\ \mathrm{Im}\int_0^z (\zeta+c)(x'+\alpha)(y'+\beta)\,\mathrm{d}\zeta &= \frac{(xy)^2}{2} + (\alpha\beta + xy)(\alpha y + \beta x) + \alpha\beta xy + \frac{(\alpha y + \beta x)^2}{2}. \end{aligned}$$

Then,

$$Z^{(1)}(a,0;z) = \frac{\sqrt{(x+\alpha)(y+\beta)}}{2}\left(\lambda\left(\frac{x^2-y^2}{2}+\alpha x-\beta y\right)+\mu\ln\left(\frac{\alpha\beta}{(x+\alpha)(y+\beta)}\right)\right)$$
$$+\frac{\mathrm{i}}{2\sqrt{(x+\alpha)(y+\beta)}}\left(\lambda\left(\frac{(xy)^2}{2}+(\alpha\beta+xy)(\alpha y+\beta x)+\alpha\beta xy+\frac{(\alpha y+\beta x)^2}{2}\right)\right.$$
$$\left.+\mu\left(\frac{x^2-y^2}{2}+\alpha x-\beta y\right)\right).$$

One can check that for any λ and μ the real part of this function is indeed a solution of (14) with the potential (37). Substituting λ and μ from (38) or (39) we obtain $Z^{(1)}(1,0;z)$ and $Z^{(1)}(\mathrm{i},0;z)$, respectively.

5. Complex potentials

Our approach can also be applied to the Schrödinger equation (14) with ν being a complex function, though in this case complex numbers become insufficient, and one should consider the bicomplex generalization of the pseudoanalytic function theory.

Together with the imaginary unit i let us consider another imaginary unit j such that $\mathrm{j}^2 = -1$ and $\mathrm{ij} = \mathrm{ji}$. Bicomplex numbers have the form $z_1 + z_2\mathrm{i}$ where z_1 and z_2 can be considered as complex with respect to the unit j: $z_{1,2} = x_{1,2} + \mathrm{j}y_{1,2}$, $x_{1,2}$ and $y_{1,2}$ being real numbers. Bicomplex numbers obviously form a commutative algebra which contains a subset of zero divisors.

Now we assume that $\nu = \nu_1 + \mathrm{j}\nu_2$ where ν_1 and ν_2 are real-valued functions. The factorization (15) remains valid for $\varphi = \varphi_1 + \mathrm{j}\varphi_2$ as well as all the results of the present work.

6. Conclusions

In this work, we considered the real stationary two-dimensional Schrödinger equation. With the aid of any of its particular solution, we construct the Vekua equation possessing the following special property. The real parts of its solutions are solutions of the original Schrödinger equation and the imaginary parts are solutions of an associated Schrödinger equation with a potential having the form of a potential obtained after the Darboux transformation. After having applied Bers' approach to this Vekua equation we obtained a locally complete system of solutions of the original Schrödinger equation which can be constructed explicitly for an ample class of Schrödinger equations, namely when the Schrödinger equation admits a particular solution satisfying the proposed Condition S. We established that in such special cases as of the potential being a function of one Cartesian, spherical, parabolic or elliptic variable the condition is fulfilled. We gave some examples of application of the proposed procedure for obtaining a locally complete system of solutions of the Schrödinger equation. The procedure is algorithmically simple and can be implemented with the aid of a computer system of symbolic or numerical calculation. The obtained system of solutions is a good candidate for numerical analysis of boundary value problems for the Schrödinger equation.

Acknowledgment

The author wishes to express his gratitude to CONACYT for supporting this work via the research project 43432.

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